

Lyapunov Functions for Pullback Attractors: Past-to-Present Stability, Construction Recipes, and a $q \rightarrow \tau(t)$ Perspective

Douglas C. Youvan

doug@youvan.com

www.youvan.ai

October 31, 2025

Classical Lyapunov theory certifies stability for autonomous systems by exhibiting an energy-like function that decreases along trajectories. For nonautonomous dynamics—where laws or forcings vary in time—the appropriate long-time object is a time-indexed family: the pullback attractor. Its pragmatic stance is to judge present stability using only information from the past up to now, without presuming anything about the future. This “past $\rightarrow$ present” view is central to Kloeden’s program, which provides a converse-style Lyapunov theorem for pullback attraction: under mild hypotheses, one can construct a Lyapunov function $V(p, x)$ that (i) separates states from the present attractor slice A_p , (ii) decays along pullbacked trajectories, and (iii) satisfies forward exponential estimates even when forward attraction is absent. This paper synthesizes the pullback framework and Kloeden’s Lyapunov construction into a practical recipe. We formalize the cocycle setting, state assumptions, and derive a weighted-distance form of V from absorbing neighborhoods. We give implementable diagnostics, illustrate with time-varying ODEs and sketches for driven dissipative PDEs, and compare pullback, forward, and uniform attractors. Finally, we interpret V and A_p through the informational alignment lens $q \rightarrow \tau(t)$: V becomes an operational divergence to “what is stably real now,” defined entirely by realized history, enabling certification of alignment without forecasting.

Keywords: pullback attractor, nonautonomous dynamical system, cocycle, Lyapunov function, absorbing neighborhood, random attractor, input-to-state stability, nonstationary forcing, $q \rightarrow \tau(t)$, informational alignment.

A collaboration with GPT-5. CC4.0.

Outline

1. Nonautonomous Stability in One Page
 - 1.1 Why forward attractors can mislead under drift
 - 1.2 Pullback logic: evaluate with realized past only
 - 1.3 Aims: constructive V and practical diagnostics
2. Setting and Preliminaries
 - 2.1 Base flow and cocycle: $\theta_t, \Phi(t, p, x)$
 - 2.2 Pullback absorption and pullback attractor A_p
 - 2.3 Metrics, semidistances, compactness hypotheses
3. Kloeden's Lyapunov Theorem for Pullback Attractors
 - 3.1 Assumptions (continuity, local Lipschitz bounds, absorbing families)
 - 3.2 Constructing $V(p, x)$ from weighted distances to $\Phi(t, \theta_{-t}p, B_{\theta_{-t}p})$
 - 3.3 Properties: separation, pullback decay, forward exponential estimate
 - 3.4 Stability without foreknowledge of the future
4. Computable Recipe and Diagnostics
 - 4.1 Estimating absorbing neighborhoods (bounds or data-driven)
 - 4.2 Discretizing the driver and computing V numerically
 - 4.3 Present-time maps: level sets of $V(p, \cdot)$
 - 4.4 Robustness to noise and parameter drift
5. Worked Examples
 - 5.1 Scalar time-varying ODE: pullback vs forward behavior
 - 5.2 Quasilinear parabolic PDE with seasonal forcing (sketch)
 - 5.3 Counterexample intuition: forward instability yet pullback attraction
6. Relations and Extensions
 - 6.1 Forward, uniform, and pullback attractors: what each certifies
 - 6.2 Random dynamical systems: pullback $\rightarrow$ random attractors
 - 6.3 ISS-style bounds for time-varying inputs
 - 6.4 Comparison with Lyapunov–Perron and exponential dichotomies
7. $q \rightarrow \tau(t)$ Interpretation
 - 7.1 Present “truth slice” $\tau(t) \equiv A_{\theta_t}p$
 - 7.2 V as operational divergence $D(q(t) \parallel \tau(t))$ computable from history
 - 7.3 Alignment without prophecy
 - 7.4 Controllers to keep V small under unknown futures

- 8. Applications and Case Sketches
 - 8.1 Climate/earth systems
 - 8.2 Epidemiology and operations
 - 8.3 Finance and policy regimes
- 9. Practical Guidelines and Pitfalls
 - 9.1 Verifying absorption vs mistaking transients
 - 9.2 Finite-window surrogates for infinite-horizon pullback
 - 9.3 When forward attractors remain the right tool
- 10. Conclusion
 - 10.1 What pullback Lyapunov functions certify—and what they do not
 - 10.2 The virtue of “past → present” realism
 - 10.3 Next steps: numerics, data-driven drivers, open benchmarks

References

1. Nonautonomous Stability in One Page

1.1 Why forward attractors can mislead under drift

“Forward” attraction asks whether trajectories at time t converge, as $t \rightarrow \infty$, to some set A (often time-independent). In nonautonomous worlds the law itself drifts: parameters, forcings, regimes change. Then a forward attractor answers a question about an unknown future driver, not about the system you actually ran. Two quick intuitions:

- Scenario dependence. Take $x' = a(t)x - x^3$ with $a(t)$ slowly drifting. Today’s state may look stable if you freeze $a(t)$ at its current value; yet if $a(t)$ will cross 0 later, the long-run set changes. Forward attraction silently bakes in that future crossing.
- One-shock paradox. For $x' = -x + u(t)$, let $u(t)=0$ for all past times and $u(t)=1$ for all future times after tomorrow. Forward analysis judges convergence to $x=1$; but this conclusion depends entirely on tomorrow’s step, which you have not observed.

Hence, a forward attractor can be correct but irrelevant to “is the system aligned now?” when the driver keeps evolving.

1.2 Pullback logic: evaluate with realized past only

Pullback attraction answers a different question: given the actual history up to the present base point p (think p encodes the realized driver), do trajectories initialized far in the past and evolved along that history land near a present-time slice A_p ? Formally, with a cocycle Φ and base flow θ , pullback attraction to A_p means

$$\text{dist}(\Phi(t, \theta_{\{-t\}}p, x), A_p) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

This reframes stability as past $\rightarrow$ present: no prophecy required. You certify what is stably real now, conditional on the path that actually happened. It also cleanly separates notions:

- Pullback attraction can hold even when forward attraction fails (future may drift elsewhere).
- The attractor itself is a family $\{A_p\}$: different presents have different stable slices.

1.3 Aims: constructive V and practical diagnostics

Our aims are operational:

(A) Constructive Lyapunov witness $V(p, x)$.

We build a nonnegative function V with:

- $V(p, x)=0$ iff $x \in A_p$ (separation),
- $V(\theta_{-t} p, \Phi(t, \theta_{\{-t\}}p, x)) \rightarrow 0$ as $t \rightarrow \infty$ (pullback decay),

- and typically a forward estimate $V(\theta_t p, \Phi(t, p, x)) \leq e^{-ct} V(p, x)$ for $t \geq 0$ (a controlled decrease without assuming forward attraction).

The construction uses absorbing neighborhoods B_p and weighted distances to the sets $\Phi(t, \theta_{-t} p, B_{\theta_{-t} p})$, with weights tuned by local Lipschitz bounds.

(B) Practical diagnostics you can compute “today.”

- Level-set maps: plot $x \mapsto V(p, x)$ at the present p to visualize how far states are from A_p .
- Finite-window pullback proxy: approximate V by maximizing over t in $[0, T]$ (T large enough for absorption) to avoid infinite horizons.
- Absorption check: verify that $\Phi(t, \theta_{-t} p, U)$ enters B_p for large t for any bounded U ; this guards against mistaking long transients for attraction.
- Drift sensitivity: evaluate V under slightly perturbed recent histories to estimate robustness margins.
- Controller/readiness metric: use V as an alignment gauge for $q \rightarrow \tau(t)$: keeping V small means q is practically aligned with the present slice $\tau(t) = A_p$, based solely on realized history.

Bottom line: forward attractors answer “where might we go if the future behaved like X ,” while pullback Lyapunov functions answer “how aligned are we now given the path that actually occurred”—and they do so with a computable $V(p, x)$.

2. Setting and Preliminaries

2.1 Base flow and cocycle: $\theta_t, \Phi(t, p, x)$

Base space. Let (P, d_P) be a metric “environment” space that encodes the realized driver (e.g., a forcing signal, parameter path, or symbol sequence).

Base flow. A family $\{\theta_t\}_{t \in \mathbb{R}}$ of homeomorphisms $\theta_t: P \rightarrow P$ is a (continuous) flow if

- $\theta_0 = \text{id}$,
- $\theta_{t+s} = \theta_t \circ \theta_s$,
- $(t, p) \mapsto \theta_t p$ is continuous.

Think of p as “now” and $\theta_{-t} p$ as “the base point t units in the past along the realized history.”

State space. Let (X, d_X) be a complete metric space (Banach spaces and Hilbert spaces are typical).

Cocycle / process. A (nonautonomous) cocycle over θ is a map

$$\Phi: [0, \infty) \times P \times X \rightarrow X, (t, p, x) \mapsto \Phi(t, p, x),$$

that is continuous, with $\Phi(0, p, \cdot) = \text{id}_X$ and the cocycle identity

$$\Phi(t+s, p, x) = \Phi(t, \theta_s p, \Phi(s, p, x)) \text{ for all } t, s \geq 0.$$

Equivalently, in “process” notation one writes $\Phi(t, s, p)x$ for evolution from time s to t and relates it via $p_s = \theta_s p_0$ and $\Phi(t, s, p_0)x = \Phi(t-s, p_s, x)$.

Minimal regularity. We will assume:

- (R1) For each bounded $K \subset X$ and compact $Q \subset P$, $\Phi(\cdot, \cdot, \cdot)$ is jointly continuous on $[0, T] \times Q \times K$.
- (R2) Local Lipschitz in x on bounded sets: there is $L(p, T, K)$ with $d_X(\Phi(t, p, x), \Phi(t, p, y)) \leq e^{L(p, T, K)t} d_X(x, y)$ for $t \in [0, T]$, p in compact sets, x, y in K .
These are typical for time-dependent ODEs/PDEs generating continuous processes.

2.2 Pullback absorption and pullback attractor A_p

Pullback absorbing family. A family of bounded sets $\{B_p \subset X\}_{p \in P}$ is *pullback absorbing* if for every bounded $U \subset X$ and every $p \in P$ there exists $T = T(U, p)$ such that for all $t \geq T$, $\Phi(t, \theta_{-t} p, U) \subset B_p$.

Intuition: start far enough in the past along the realized base point p , evolve U forward by the actual history; eventually you enter the “present-time” absorbing set B_p .

Asymptotic compactness (pullback). The cocycle is pullback asymptotically compact if for every p and bounded U , any sequence $t_n \rightarrow \infty$ and $x_n \in U$ yields a precompact sequence $\{\Phi(t_n, \theta_{-t_n} p, x_n)\}$ in X . This rules out mass “leaking to infinity” in the pullback limit.

Pullback ω -limit and attractor. Given a pullback absorbing $\{B_p\}$, define the pullback ω -limit at p : $\omega_p(B) := \bigcap \{\tau \geq 0 \mid \bigcup \{t \geq \tau\} \Phi(t, \theta_{-t} p, B_{\theta_{-t} p}) \neq \emptyset\}$.

If pullback asymptotic compactness holds, each $\omega_p(B)$ is nonempty, compact, and independent of the particular absorbing family (i.e., canonical).

A family $\{A_p\}_{p \in P}$ is a pullback attractor if:

1. Compactness: each A_p is compact and $A = \{A_p\}$ is bounded in an appropriate uniform sense on compact subsets of P ;
2. Invariance (cocycle-invariance): $\Phi(t, p, A_p) = A_{\theta_t p}$ for all $t \geq 0$;
3. Pullback attraction: for every bounded $U \subset X$, $\text{dist}_X(\Phi(t, \theta_{-t} p, U), A_p) \rightarrow 0$ as $t \rightarrow \infty$.

Under the conditions above, the canonical choice is $A_p = \omega_p(B)$. Uniqueness holds within the class of compact families satisfying (2)–(3).

Remark (forward vs pullback). Forward attraction would ask $\text{dist}_X(\Phi(t,p,U), A_{\{\theta_t p\}}) \rightarrow 0$ as $t \rightarrow \infty$, which can fail in drifting environments even when pullback attraction holds. The pullback notion conditions on the realized past, matching the “no prophecy” stance.

2.3 Metrics, semidistances, compactness hypotheses

Point–set distance. For $x \in X$ and a nonempty $S \subset X$:

$$\text{dist}_X(x, S) := \inf_{y \in S} d_X(x, y).$$

Hausdorff semidistance. For nonempty bounded sets $A, B \subset X$ define the one-sided semidistance

$$\rho(A, B) := \sup_{a \in A} \text{dist}_X(a, B).$$

This measures how well A is covered by B but is not symmetric. The Hausdorff distance is

$$d_H(A, B) := \max\{\rho(A, B), \rho(B, A)\}.$$

Pullback attraction is commonly stated via the semidistance: $\rho(\Phi(t, \theta_{-t}p, U), A_p) \rightarrow 0$.

Uniform boundedness on P . When P is noncompact, one typically asserts boundedness “locally uniformly” in p : for each compact $Q \subset P$ there exists a bounded set $K_Q \subset X$ with $A_p \subset K_Q$ for all $p \in Q$, and likewise for absorbing families $\{B_p\}$.

Precompactness criteria. To ensure asymptotic compactness, one of the following is used (depending on the model):

- Compact embedding: flows generated by parabolic PDEs often yield compactness via smoothing and compact embeddings (Aubin–Lions).
- Ball/energy estimates: bounded absorbing balls in a reflexive Banach space plus asymptotic smoothness (e.g., uniform Lipschitz+Kolmogorov compactness).
- Tail estimates: for unbounded domains, prove decay of “tails” and compactness on bounded subdomains.

Continuity in p . For regular dependence on the base point, assume:

- (C1) $p \mapsto B_p$ is upper semicontinuous in the Hausdorff topology on bounded sets;
- (C2) $(t, p, x) \mapsto \Phi(t, p, x)$ is continuous.

Then $p \mapsto A_p$ is upper semicontinuous (and often measurable), which is important for parameter studies and numerics.

Minimal existence theorem (pullback).

If (i) Φ is pullback asymptotically compact, and (ii) there exists a pullback absorbing family $\{B_p\}$, then the pullback attractor $\{A_p\}$ exists, is unique, invariant, and attracts all bounded U in the pullback sense. Moreover, $A_p = \omega_p(B)$ for any absorbing family B .

Skew-product viewpoint (optional). Define the skew-product flow on $P \times X$ by

$$\Psi_t(p, x) := (\theta_t p, \Phi(t, p, x)).$$

Invariance of $\{A_p\}$ is equivalent to invariance of the global set $\mathfrak{A} := \{(p, x) : x \in A_p\}$ under Ψ_t .

This viewpoint makes compactness proofs and Lyapunov constructions (Section 3) more systematic.

What you need to carry forward.

- θ organizes the realized history; Φ is the evolution driven by that history.
- A pullback absorbing family and pullback asymptotic compactness give you a canonical $\{A_p\}$.
- Use the semidistance $p(\cdot, \cdot)$ to state attraction, and keep an eye on continuity/compactness-in- p to make diagnostics stable.
These ingredients are precisely what the constructive Lyapunov function $V(p, x)$ (next section) will exploit.

3. Kloeden's Lyapunov Theorem for Pullback Attractors

3.1 Assumptions (continuity, local Lipschitz bounds, absorbing families)

Work in a cocycle setting over a base flow θ :

- State space: (X, d) complete metric (Banach/Hilbert typical).
- Base space: (P, d_P) with continuous flow $\theta_t: P \rightarrow P$.
- Cocycle: $\Phi: [0, \infty) \times P \times X \rightarrow X$, continuous, $\Phi(0, p, \cdot) = \text{id}$, and $\Phi(t+s, p, \cdot) = \Phi(t, \theta_s p, \Phi(s, p, \cdot))$.

Regularity.

1. (Joint continuity) Φ is continuous on bounded time–base–state blocks.
2. (Local Lipschitz in x) For every compact $Q \subset P$ and bounded $K \subset X$ there exists $L_Q, K \geq 0$ so that

$$d(\Phi(t,p,x), \Phi(t,p,y)) \leq e^{\int_0^t L_Q(s,p) ds} d(x,y) \text{ for all } p \in Q, x,y \in K, t \in [0,T].$$

(For ODEs: L_Q, K can come from a local Lipschitz bound on $f(\theta_t p, \cdot)$.)

Absorption and asymptotic compactness.

3) (Pullback absorbing family) There exists $\{B_p\}_{p \in P}$ bounded such that for each bounded $U \subset X$ and p , $\exists T(U,p)$ with $\Phi(t, \theta_{-t}p, U) \subset B_p$ for all $t \geq T$.

4) (Pullback asymptotic compactness) For any p , bounded U , $t_n \rightarrow \infty$, $x_n \in U$, the sequence $\Phi(t_n, \theta_{-t_n}p, x_n)$ is precompact in X .

Under (3)–(4) the pullback attractor A_p exists, is invariant, compact, and attracts bounded sets in the pullback sense.

3.2 Constructing $V(p,x)$ from weighted distances to $\Phi(t, \theta_{-t}p, B_{\theta_{-t}p})$

Fix a (locally uniform) Lipschitz envelope $L(p) \geq 0$ for Φ on the relevant bounded sets (e.g., $L(p) = L_{Q,K}$ for p in a compact Q). Let $\alpha > 0$ be a chosen decay rate (often $\alpha = 1$ after rescaling time). Define the weight

$$W(p,t) := \exp(-\alpha t - \int_0^t L(\theta_{-s}p) ds).$$

Define the reference image at horizon t :

$$S(p,t) := \Phi(t, \theta_{-t}p, B_{\theta_{-t}p}) \subset X.$$

Define the pullback Lyapunov candidate

$$V(p,x) := \sup_{t \geq 0} W(p,t) \cdot \text{dist}(\Phi(t, \theta_{-t}p, x), S(p,t)).$$

Remarks:

- Intuition: at each horizon t , compare the pullback trajectory of x to the concurrent image of the absorbing set; weight that distance by $W(p,t)$ that compensates local expansion (via L) and enforces net decay (via α).
- The sup over $t \geq 0$ folds in the “worst residual misalignment” across pullback horizons; absorption ensures the sup is finite on bounded sets.
- One may use equivalent formulations, e.g., distances to A_p in place of $S(p,t)$, once existence of A_p is known. The $S(p,t)$ -form is constructive: it needs only Φ and B .

Implementation notes:

- In numerics, approximate V by $V_T(p,x) = \max_{t \in [0,T]} W(p,t) \cdot \text{dist}(\cdot, \cdot)$ with T large enough to pass absorption (Section 4).

3.3 Properties: separation, pullback decay, forward exponential estimate

(P1) Nonnegativity and separation.

$V(p,x) \geq 0$. Moreover, $V(p,x)=0$ iff $x \in A_p$.

Sketch. If $x \in A_p$, then for each t , $\Phi(t, \theta_{-t}p, x) \in S(p,t)$ by invariance of A , hence each term in the sup is 0. Conversely, if $V(p,x)=0$, then for every t the distance from $\Phi(t, \theta_{-t}p, x)$ to $S(p,t)$ is 0; asymptotic compactness + absorption yield that the pullback ω -limit of x at p lies in A_p , whence $x \in A_p$ by invariance.

(P2) Pullback decay (along past $\rightarrow$ present trajectories).

For any $x \in X$ and $p \in P$,

$V(\theta_{-t}p, \Phi(t, \theta_{-t}p, x)) \rightarrow 0$ as $t \rightarrow \infty$.

Sketch. Evaluate V at the shifted base point $\theta_{-t}p$ and the pullbacked state. After reindexing the sup (use the cocycle property), each horizon in the new sup corresponds to a larger horizon in the old sup, and the weights $W(\theta_{-t}p, \cdot)$ inherit an extra factor $e^{-\alpha t}$. Absorption ensures the raw distances in the tail are controlled, hence the overall expression vanishes.

(P3) Forward exponential estimate (without assuming forward attraction).

For any $t \geq 0$,

$V(\theta_{-t}p, \Phi(t, p, x)) \leq e^{-\alpha t} V(p,x)$.

Sketch. Use the cocycle identity to align horizons and note:

- the distance term is nonexpansive under “best matching” with $S(\cdot, \cdot)$,
- the weight multiplies by $e^{-\alpha t}$ and cancels the accumulated local Lipschitz growth via the integral of L along $\theta_{-s}p$.

Taking the sup over horizons yields the inequality.

(P4) Local equivalence to distance to A_p .

On bounded subsets where L and absorbing radii are uniformly controlled, there exist class- K functions k_1, k_2 (increasing, $k_i(0)=0$) such that

$k_1(\text{dist}(x, A_p)) \leq V(p,x) \leq k_2(\text{dist}(x, A_p))$.

Thus V metrizes nearness to A_p up to monotone transforms.

Consequences:

- V is a bona fide Lyapunov witness for the pullback notion: it decreases along forward time with a guaranteed exponential factor (P3) and vanishes along pullbacked trajectories (P2).

- Crucially, (P3) does not assert forward attraction to $A_{\{\theta_t p\}}$; it only controls the Lyapunov scale. Forward attraction may fail under drift.

3.4 Stability without foreknowledge of the future

The construction uses only the realized history embedded in $\theta_{[-s]p}$ (through $S(p,t)$ and the weight integral). No assumption about how $\theta_t p$ will evolve for $t>0$ is needed:

- Certification “now.” If $V(p,x)$ is small, x is close to A_p and will remain Lyapunov-decreasing at least at rate $e^{-\alpha t}$ for as long as the same driver is applied—yet no claim is made about the location of $A_{\{\theta_t p\}}$ in the future.
- Separation of notions. You can have $V(\theta_t p, \Phi(t,p,x)) \rightarrow 0$ (exponentially), while $\Phi(t,p,x)$ does not converge to $A_{\{\theta_t p\}}$ because the future attractor slices keep drifting. V quantifies alignment with the present slice A_p , not convergence to a moving target you haven’t specified.
- Operational payoff. In applications with nonstationarity (seasonality, policy shifts, regime changes), V gives a computable “distance-to-being-stable-now,” built from pullback absorption and local Lipschitz geometry—no prophecy required.

Rule of thumb.

- Use $V(p,x)$ to answer: “How aligned are we with what is stably real now (A_p) given the history that actually occurred?”
- Use forward/uniform attractors only when the driver is stationary or when a specific future scenario is fixed in advance.

(Next, we can instantiate this for a concrete time-varying ODE and show how to estimate $L(p)$, build B_p , and compute a finite-horizon V_T numerically.)

4. Computable Recipe and Diagnostics

4.1 Estimating absorbing neighborhoods (bounds or data-driven)

Goal. Build a present-time bounded set B_p such that for any bounded U and large t , $\Phi(t, \theta_{-t}p, U) \subset B_p$. You can do this with (A) analytic bounds or (B) data-driven surrogates.

(A) Analytic route (ODE/PDE with estimates).

- Dissipation/energy inequality. Find $E(x) \geq 0$ with $\dot{E} \leq -\gamma E + \beta(p, t)$. If $\sup_{s \leq 0} \beta(\theta_s p, s) \leq \tilde{\beta}(p)$, then any pullback trajectory satisfies $E(t) \leq e^{-\gamma t} E(0) + \tilde{\beta}(p)/\gamma$. Choose $B_p = \{x: E(x) \leq r(p)\}$ with $r(p)$ slightly larger than $\tilde{\beta}(p)/\gamma$.
- Coercivity to radius. If $E(x) \geq c \|x\|^2 - C$, convert E -balls to norm-balls.
- PDEs. Use standard a priori estimates (e.g., Grönwall + Poincaré) to get absorbing radii in H^1 (and compactness via embeddings).

(B) Data-driven route (time-varying / black-box).

- Pullback fragments. From observed driver history $\{p_{-k}\}_{k \geq 0}$, simulate pullbacked images of a coarse covering $U = \{x^{(i)}\}$ for a growing horizon T . Once the empirical radius stabilizes, set $B_p := \text{Hull}_\epsilon(\{\Phi(T_k, \theta_{-T_k} p, x^{(i)})\}_{i,k})$ (convex or α -shape hull with margin ϵ).
- Shrinking window check. Verify that enlarging the horizon (e.g., $T \rightarrow 1.5T$) no longer expands the hull by more than a tolerance. That's your empirical absorption.

Tip. In either route, keep B_p *slightly* inflated (5–15%) to buffer numerical error and unmodeled micro-drift.

4.2 Discretizing the driver and computing V numerically

We approximate

$$V(p, x) = \sup_{t \geq 0} W(p, t) \cdot \text{dist}(\Phi(t, \theta_{-t} p, x), S(p, t)), S(p, t) := \Phi(t, \theta_{-t} p, B_{\theta_{-t} p}),$$

with weight $W(p, t) = \exp(-\alpha t - \int_0^t L(\theta_{-s} p) ds)$.

Discretization choices.

- Time grid. Pick Δt and $T_{\max}$ with $T_{\max}$ beyond the absorption time observed in 4.1. Use $t_j = j\Delta t$.
- Driver history. Store $p_{-j} := \theta_{-t_j} p$. For data-based drivers, p_{-j} is just the recorded signal at time $-t_j$.
- Local Lipschitz L .

- *Model-based*: $L(p_{-j}) \approx \sup_{\|v\|=1} \|D_x f(p_{-j}, x^*) v\|$ around the absorbing shell (cheap bound: operator norm of Jacobian).
- *Black-box*: estimate via finite differences: evolve two close states over Δt and regress $\log \frac{\|\delta x(t+\Delta t)\|}{\|\delta x(t)\|} / \Delta t$.

Algorithm (finite-horizon V_T).

1. Precompute $B_{p_{-j}}$ (analytic radius or empirical hull).
2. For each t_j :
 - Push forward $B_{p_{-j}}$: $S_j := \Phi(t_j, p_{-j}, B_{p_{-j}})$.
 - Evolve the test state: $x_j := \Phi(t_j, p_{-j}, x)$.
 - Compute $d_j := \text{dist}(x_j, S_j)$ (nearest neighbor to a mesh of S_j , or signed distance if available).
 - Accumulate weight $w_j := \exp(-\alpha t_j - \sum_{m=1}^j L(p_{-m}) \Delta t)$.
3. Set $V_T(p, x) := \max_{j \leq T_{\max}/\Delta t} w_j d_j$.
4. Increase $T_{\max}$ until V_T stabilizes within tolerance; call the result $V(p, x)$.

Complexity notes.

- If $B_{p_{-j}}$ is a ball and Φ is smooth, you can approximate S_j by pushing forward the ball center and linearized image of the radius (ellipsoid), which makes distance queries $O(1)$.
- Otherwise, sample $B_{p_{-j}}$ with M points; then distance is $O(M)$ per j or $O(\log M)$ with a k-d tree.

4.3 Present-time maps: level sets of $V(p, \cdot)$

Why. $V(p, \cdot)$ is your “alignment radar” at the present base point p : low contours $\approx$ near the present attractor slice A_p .

How to plot.

- Low-dim state (2–3D). Grid the region of interest, compute V (or V_T) on grid points, and draw contours/isosurfaces.
- Mid-dim state (5–50D).

- Plot slices: fix all but 2 coordinates around a nominal state (e.g., absorber center), map the 2D slice.
 - Or plot the *ranked list* of candidate states by V (useful in control/operations).
- High-dim state (PDE). Project to a few physical summary features (energies, principal components) and map V over those features.

What to look for.

- Basins. Star-shaped or funnel-like low- V regions indicate robust pullback attraction. Fragmented low- V pockets may reflect multi-regime behavior.
- Sensitivity ridges. Sharp V gradients flag directions where tiny perturbations leave the absorbing corridor; monitor these for operations.
- $q \rightarrow \tau(t)$ view. Treat $\tau(t) \equiv A_p$. Plotting $V(p, \cdot)$ visualizes how “close” candidate q states are to τ now, using only realized history.

4.4 Robustness to noise and parameter drift

Noise in state and driver.

- State noise (sensor). Because V is locally equivalent to $\text{dist}(\cdot, A_p)$, additive state noise η perturbs V by $\approx \text{Lip}(V) \|\eta\|$. Empirically estimate $\text{Lip}(V)$ by finite differences on the map in 4.3.
- Driver noise (history). Recompute V with jittered recent history windows $\{p_{-j} + \delta p_{-j}\}_{j \leq J}$ for a small J (last few steps) to form an ensemble V -band $[V_{\min}, V_{\max}]$. A tight band indicates robustness to recent measurement error.

Drift and regime changes.

- Sliding-window L and B . Re-estimate $L(p)$ and B_p over a trailing window (e.g., last H units of time). This maintains validity when local expansion rates or dissipation change.
- Horizon stress-test. Compare V_T over increasing T . If V_T keeps changing materially past the observed absorption time, your B_p is under-inflated or true absorption hasn't occurred—pause certification.
- Counterfactual micro-futures (no prophecy). You *can* probe near-term robustness (e.g., 1–2 steps) by appending a few bounded driver perturbations p_{+1}, p_{+2} and checking the forward estimate

$$V(\theta_{\Delta} p, \Phi(\Delta, p, x)) \leq e^{-\alpha \Delta} V(p, x).$$

If this bound is loose under small, plausible shifts, flag the operating point as fragile.

Uncertainty quantification.

- Bootstrap absorbers. Rebuild B_p from resampled pullback fragments to get a distribution of radii; propagate into a band for V .
- Lipschitz envelopes. Use a conservative upper envelope $\hat{L} \geq L$. If V is still small with $\hat{L}$, your certificate is robust.

Operational thresholds.

- Define V^* (green), V^{**} (yellow), V^{***} (red) based on distance-equivalence constants in your domain. Use them as nowcasting alarms: green = aligned with τ ; yellow = keep watch; red = re-enter absorber (control action).

Quick checklist (to implement tomorrow)

1. Choose $\Delta t, T_{\max}$. Record p_{-j} for $j\Delta t \leq T_{\max}$.
2. Build $B_{p_{-j}}$ analytically or by empirical hulls; inflate $+\epsilon$.
3. Estimate $L(p_{-j})$; set $\alpha > 0$ (e.g., 1).
4. For each t_j : compute S_j , evolve $x \mapsto x_j$, get d_j, w_j .
5. Output $V_T(p, x) = \max_j w_j d_j$; increase $T_{\max}$ until stable.
6. Plot $V(p, \cdot)$ level sets; mark thresholds (green/yellow/red).
7. Run robustness: jitter last J driver steps, bootstrap B_p , inflate L .
8. Use V as an on-line gauge of $q \rightarrow \tau(t)$ alignment—built entirely from realized history.

5. Worked Examples

5.1 Scalar time-varying ODE: pullback vs forward behavior

Model A (drifting linear growth):

$$x' = 2t \cdot x.$$

The solution from time s to t is $x(t) = x(s) \cdot \exp(t^2 - s^2)$.

- Pullback view (present $t = 0$). Start at $s = -T$ with any bounded $x(-T)$. Then $x(0) = x(-T) \cdot \exp(-T^2) \rightarrow 0$ as $T \rightarrow \infty$.
Hence the pullback attractor slice at “now” is $A_p = \{0\}$. Every bounded set pulled forward from the distant past collapses to 0 at $t = 0$.
- Forward view. From any $x(0) \neq 0$, $x(t) = x(0) \cdot \exp(t^2) \rightarrow \infty$ as $t \rightarrow \infty$.
There is no forward attraction to 0.
Lesson. Perfect illustration of “stable now by history” vs “no guarantee for tomorrow.”
Pullback attraction holds; forward attraction fails.

Model B (slowly drifting pitchfork):

$x' = a(t)x - x^3$, with $a(t)$ increasing through 0 (e.g., $a(t) = \tanh t$).

- For $t \ll 0$ the unique stable state is $x=0$; for $t \gg 0$ the equilibria $\pm\sqrt{a(t)}$ stabilize and $x=0$ is unstable.
- Pullback: Fix the present time t_0 . The pullback slice A_p at t_0 is what is stable given the realized past up to t_0 . If $t_0 < 0$, $A_p \approx \{0\}$; if $t_0 > 0$ but not too large, the pullback slice can still be near 0 because much of the past enforced contraction to 0 before the bifurcation completed.
- Forward: As t increases further, trajectories move toward $\pm\sqrt{a(t)}$, i.e., the *future* equilibria.
Lesson. Forward conclusions depend on unobserved future drift; pullback certifies the present slice using only history.

5.2 Quasilinear parabolic PDE with seasonal forcing (sketch)

Consider a prototypical diffusion–reaction equation on a bounded domain Ω with smooth boundary and homogeneous Dirichlet data:

$$u_t - \nabla \cdot (\kappa(x) \nabla u) + c(t)u + g(u) = f(t, x), \quad x \in \Omega, \quad t \in \mathbb{R},$$

with $\kappa(x)$ uniformly elliptic, $c(t) \geq c_0 > 0$, and g locally Lipschitz with $g(u)u \geq 0$ (dissipative). Let the driver be $p = (c(\cdot), f(\cdot, \cdot))$ living on a compact hull under time-shift θ_t .

Absorbing estimate (energy route).

Multiply by u and integrate:

$$(1/2) \frac{d}{dt} \|u\|^2 + \lambda \|\nabla u\|^2 + c(t) \|u\|^2 + \langle g(u), u \rangle = \langle f(t), u \rangle.$$

Using $c(t) \geq c_0$, $g(u)u \geq 0$, Poincaré ($\|u\|^2 \leq C_P \|\nabla u\|^2$), and Young’s inequality:

$$\frac{d}{dt} \|u\|^2 \leq -\gamma \|u\|^2 + \beta(t),$$

with $\gamma > 0$ and $\beta(t) = C \|f(t)\|^2$. Grönwall in pullback form gives, for history ending at time τ :

$$\|u(\tau)\|^2 \leq e^{\{-\gamma(\tau-s)\}} \|u(s)\|^2 + \int_{-\infty}^{\tau} e^{\{-\gamma(\tau-\xi)\}} \beta(\xi) d\xi.$$

Let $\tilde{\beta}(p, \tau) := \sup_{r \leq \tau} \int_{-\infty}^r e^{-\gamma(r-\xi)} \beta(\xi) d\xi$. Then choose the absorbing ball

$$B_{\{\theta_{-\tau} p\}} := \{ u : \|u\|^2 \leq (1+\epsilon) \cdot \bar{\beta}(p, \tau) \}.$$

This is a *present-time* bound built only from past forcing. Standard compactness (Aubin–Lions) gives pullback asymptotic compactness, hence a pullback attractor slice $A_{\{\theta_{-\tau} p\}} \subset H^1_o(\Omega)$.

Computable V (sketch).

With local Lipschitz envelope $L(p)$ for the process in H , define the weight

$$W(p, t) = \exp(-\alpha t - \int_0^t L(\theta_{-s} p) ds),$$

$$S(p, t) = \Phi(t, \theta_{-t} p, B_{\{\theta_{-t} p\}}), \text{ and}$$

$$V(p, u) = \sup_{t \geq 0} W(p, t) \cdot \text{dist}(\Phi(t, \theta_{-t} p, u), S(p, t)).$$

In practice one simulates Φ over one seasonal cycle (or several) to approximate $S(p, t)$, then evaluates V on a reduced basis (POD/EOFs) of the state. Level sets of $V(p, \cdot)$ visualize the “now-stable” regime under the realized seasonal history.

Takeaway. Even with seasonal drift, the present slice A_p is well defined and certifiable. Forward attraction may fail if next season differs; pullback remains meaningful and computable.

5.3 Counterexample intuition: forward instability yet pullback attraction

Beyond Model A, here are two intuitive patterns where pullback says “safe now,” forward says “don’t count on it.”

(i) Transiently contracting, ultimately expanding.

$x' = -\sigma(t) x$ with $\sigma(t) > 0$ for $t \leq 0$ but $\sigma(t) < 0$ for large $t > 0$ (net expansion ahead).

- Pullback: From far past to now, contraction wins $\Rightarrow A_p = \{0\}$.
- Forward: Soon after now, $\sigma(t)$ flips sign; solutions move away from 0.
Moral: The present slice is $\{0\}$, yet forward behavior is unstable due to future expansion.

(ii) Moving target with bounded speed (chasing equilibria).

$x' = -a(x - m(t))$, with $a > 0$ and $m(t)$ a slowly drifting mean (e.g., a bounded C^1 function).

- Pullback: At time t_0 , the slice $A_p = \{m(t_0)\}$; histories bring states near $m(t_0)$.
- Forward: If $m(t)$ keeps drifting, trajectories never *converge* to $A_{\{\theta_{-t} p\}}$ (which itself moves). Nevertheless, the Lyapunov estimate holds:
 $V(\theta_{-t} p, \Phi(t, p, x)) \leq e^{-\alpha t} V(p, x)$,
so the Lyapunov scale decays even though the target keeps sliding.

Moral: V measures alignment with the present equilibrium ($\tau(t_0)=A_p$), not convergence to a perpetually moving future target.

What these examples collectively show

- Pullback attraction is a statement about the realized history culminating in “now.”
- A single scalar ODE ($x' = 2t x$) already separates pullback from forward notions cleanly.
- In dissipative PDEs with time-dependent forcing, absorbing balls and compactness give you A_p and a computable V .
- Counterexamples remind us: a small $V(p, x)$ is a certificate of alignment with $\tau(t)$ now, not a promise about where τ will be tomorrow.

6. Relations and Extensions

6.1 Forward, uniform, and pullback attractors: what each certifies

- Forward attractor (time-varying target).
A family $\{A_{\theta_t p}\}_{t \geq 0}$ such that for each bounded U ,

$$\text{dist}(\Phi(t, p, U), A_{\theta_t p}) \rightarrow 0 (t \rightarrow \infty).$$

Certifies: convergence under a fixed future driver starting now.

Fragility: depends on how the driver evolves after now; drifting or nonstationary futures can break it.

- Uniform (global) attractor (autonomous/semi-autonomous summaries).
A compact set $\mathcal{A} \subset X$ attracting all bounded sets uniformly in time/drivers within a class. Often used when the forcing is translation-bounded or the semigroup is asymptotically autonomous.
Certifies: a time-independent compact summary of all long-time behaviors (e.g., union of complete bounded trajectories).
Use when: the environment admits uniform bounds and precompactness across time; gives a single “global picture,” but can blur time-slice detail.
- Pullback attractor (past $\rightarrow$ present).
A slice family $\{A_p\}_{p \in P}$ with

$$\text{dist}(\Phi(t, \theta_{-t} p, U), A_p) \rightarrow 0 (t \rightarrow \infty).$$

Certifies: what is stably real now given the realized history—no commitment about the unknown future.

Strength: meaningful under drift/seasonality; aligns with τ : $\tau(t) \equiv A_{\theta_t p}$.

Rule of thumb.

- Stationary/ergodic futures $\rightarrow$ forward or uniform attractors are informative.
- Drifting/unknown futures $\rightarrow$ use pullback; pair with a Lyapunov $V(p, \cdot)$ to nowcast alignment with τ .

6.2 Random dynamical systems: pullback $\rightarrow$ random attractors

Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ be a metric dynamical system (driving noise), and let $\varphi(t, \omega, x)$ be a cocycle (RDS). The random analogue of pullback attraction is:

$$\text{dist}(\varphi(t, \theta_{-t}\omega, U), A(\omega)) \rightarrow 0 \text{ as } t \rightarrow \infty \text{ for a.e. } \omega,$$

with random attractor $A(\omega)$ compact, invariant, measurable.

- Conceptual bridge. Pullback in deterministic nonautonomous systems becomes *pathwise* attraction in RDS. Each noise realization ω plays the role of the base point p .
- Lyapunov construction. The same weighted-distance recipe builds a random Lyapunov function $V(\omega, x)$, with weights integrating local Lipschitz envelopes along $\theta_{-s}\omega$. Properties mirror the deterministic case: separation $V = 0 \iff x \in A(\omega)$, pullback decay along θ -shifted paths, forward exponential estimate $V(\theta_t\omega, \varphi(t, \omega, x)) \leq e^{-\alpha t} V(\omega, x)$.
- Stationary/ergodic drivers. When θ_t is ergodic and the cocycle is dissipative, $A(\omega)$ is $\mathbb{P}$ -a.s. unique and tempered; ensemble statistics of $A(\omega)$ summarize invariant random regimes.

Takeaway. Random attractors are precisely the pullback idea applied sample-wise. All past $\rightarrow$ present logic (and V) carries over with measurability/temperedness added.

6.3 ISS-style bounds for time-varying inputs

ISS (input-to-state stability) quantifies how inputs $u(\cdot)$ bound the state. For a nonautonomous system $x' = f(\theta_t p, x, u)$, a pullback-ISS Lyapunov $V(p, x)$ yields, for suitable class- $\mathcal{K}$ functions $\alpha_1, \alpha_2, \gamma$ and $c > 0$:

- Equivalence: $\alpha_1(\text{dist}(x, A_p)) \leq V(p, x) \leq \alpha_2(\text{dist}(x, A_p))$ (locally on absorbers).
- Dissipation with input gain: along trajectories,

$$\frac{d}{dt} V(\theta_t p, x(t)) \leq -c V(\theta_t p, x(t)) + \gamma(\|u(t)\|),$$

interpreted in Dini/upper right derivatives for general cocycles.

Integrating gives the pullback-ISS estimate (for $t \geq 0$):

$$V(\theta_t p, x(t)) \leq e^{-ct} V(p, x(0)) + \int_0^t e^{-c(t-s)} \gamma(\|u(s)\|) ds.$$

Consequences (translated back to state distance via $\alpha_{1,2}$):

- Bounded-input bounded-state (BIBS). If u is bounded, V remains bounded; if $u \rightarrow 0$, then $V \rightarrow 0$ (pullback alignment restored).
- “Nowcasting with disturbance.” Use V as a live metric of misalignment to $\tau(t) = A_{\theta_t p}$ with an explicit input gain γ .

Practice. Estimate γ from Jacobians or data (gain regression). In controllers, choose inputs to keep V below thresholds (green/yellow/red bands).

6.4 Comparison with Lyapunov–Perron and exponential dichotomies

Exponential dichotomy (linear hyperbolicity).

For the linearized nonautonomous system $x' = A(t)x$, an exponential dichotomy splits state space into stable/unstable bundles with exponential bounds forward/backward in time. This structure underlies:

- Lyapunov–Perron method. Construct (un)stable/inertial manifolds as fixed points of an integral map (variation-of-constants). With small Lipschitz nonlinearity, one gets forward (or two-sided) exponential contraction to these manifolds.

How it differs/complements pullback V :

- Assumptions. Dichotomy requires uniform hyperbolicity (spectral gap) along the entire time axis; pullback V needs only dissipation + absorbing sets and asymptotic compactness along the realized past.
- What is certified. Dichotomy + Lyapunov–Perron provides geometric invariant manifolds and often forward asymptotics given the specified future driver. Pullback V provides a scalar certificate of alignment with the present slice A_p independent of future drift.
- Numerics. Manifold computations can be delicate in high dimension; computing V via weighted distances to $S(p, t) = \Phi(t, \theta_{-t}p, B_{\theta_{-t}p})$ is often cheaper and directly tied to data.
- When to use which.
 - Use Lyapunov–Perron/dichotomy when hyperbolic structure is evident and the future driver is specified or stationary; you want manifolds/normal forms.
 - Use pullback V when drivers drift or are uncertain; you want a now-valid stability certificate without foreknowledge.

Bridge result (intuition).

If a system admits an exponential dichotomy uniformly in time and the nonlinearity is small, then forward attraction to a time-varying invariant graph typically holds; the pullback attractor slices coincide with these graphs at each present time. In such cases, pullback V and Lyapunov–Perron agree on the “who is stable now” verdict—pullback V just does not demand you to assume the future.

7. $q \rightarrow \tau(t)$ Interpretation

7.1 Present “truth slice” $\tau(t) \equiv A_{\theta_t} p$

At any clock time t with realized driver state $\theta_t p$, define the present truth slice

$\tau(t) := A_{\theta_t} p$.

This is the set of states that are stably real now, certified by pullback attraction under the *actual* history up to t . As the environment drifts, the family $\{\tau(t)\}$ forms a moving foliation of “now-truth,” without assuming anything about what comes next. In your lens, $q(t) \rightarrow \tau(t)$ means “the operative model/state $q(t)$ aligns to what is presently stabilizable.”

7.2 V as operational divergence $D(q(t) \parallel \tau(t))$ computable from history

The Lyapunov witness $V(p, x)$ evaluated at the present base point $p_t := \theta_t p$ induces a data-driven divergence:

- Define $D(q(t) \parallel \tau(t)) := V(p_t, q(t))$.
- Properties:
 - $D \geq 0$, and $D = 0$ iff $q(t) \in \tau(t)$.
 - Along *pullbacked* trajectories, $D \rightarrow 0$ (historical alignment).
 - Along *forward* evolution, D decays at least exponentially,
 $V(\theta_{-\Delta} p_t, \Phi(\Delta, p_t, q(t))) \leq e^{-\alpha \Delta} V(p_t, q(t))$,
even if $\tau(t+\Delta)$ moves.
- Constructibility: D depends only on (i) the realized past through the sets $S(p_t, \cdot) = \Phi(\cdot, \theta_{-\cdot} p_t, B_{\{\theta_{-\cdot} p_t\}})$, and (ii) local Lipschitz envelopes $L(\theta_{-s} p_t)$. No forecast is used.
Consequently, D is an operational, history-conditioned misalignment—a scalar you can compute and threshold online.

7.3 Alignment without prophecy

Forward convergence asks, “Where will we land given a future we must assume?” Pullback V asks, “How aligned are we now, given only what actually happened?” Thus:

- A small D certifies present-time alignment to $\tau(t)$ without committing to any scenario.
- If τ drifts tomorrow, D today was still meaningful: it measured proximity to the real, present slice, not to a hypothetical future.
- In regimes with seasonal or adversarial drift, D functions as a nowcasting metric: detect when $q(t)$ exits the low- D corridor and trigger adaptation, rather than promise long-horizon convergence to a moving target.

7.4 Controllers to keep V small under unknown futures

Treat $V(p_t, \cdot)$ as a real-time objective and design alignment controllers that minimize D subject to actuation limits:

1. Dissipativity-shaped feedback.
Choose u to enforce a differential inequality
 $\dot{D} \leq -c D + \gamma(\|w\|)$,
where w collects disturbances/inputs you don't control. This is a pullback-ISS design:

keep $\gamma(\|w\|)$ small (filtering/robustification) and pick c by gain scheduling on the absorber shell.

2. Model predictive “no-prophecy” control (short horizon).
Optimize u over a short receding window to reduce V_T (finite-horizon proxy of V), using only the realized driver and bounded near-term uncertainties. Warm-start from the previous solution; enforce $V_T \leq V_{\text{threshold}}$ as a soft constraint.
3. Sliding-mode / corridor control.
Define a low-D corridor $C_\varepsilon := \{x : V(p_t, x) \leq \varepsilon\}$. Drive the system with u that keeps the closed-loop trajectory inside C_ε using barrier or control-Lyapunov–barrier functions based on $\partial V / \partial x$ (estimated numerically).
4. Adaptive envelopes (drift-aware).
Re-estimate $L(\theta_{-s} p_t)$ and absorber radii on a trailing window; tighten or relax gains so that the guaranteed forward estimate

$$V(\theta_{\Delta} p_t, \Phi(\Delta, p_t, q)) \leq e^{-\alpha \Delta} V(p_t, q)$$
remains valid under plausible micro-drifts.
5. Multi-objective alignment.
When $\tau(t)$ is set-valued with internal preferences (e.g., energy, cost), use $V + \lambda \cdot r(x)$ as a composite objective, where r penalizes undesirable corners of $\tau(t)$; this keeps alignment while biasing toward operationally favorable slices.

Operational playbook: monitor $D = V(p_t, q(t))$ in real time; keep it in a green band with the simplest controller that enforces $\dot{D} < 0$ locally; re-identify absorbers and Lipschitz envelopes on a sliding window; and treat excursions (D spikes) as actionable signals to re-align $q \rightarrow \tau(t)$ —all without invoking any prophecy about the future.

8. Applications and Case Sketches

8.1 Climate/earth systems

Setting. Time-varying diffusion–advection–reaction models for temperature, moisture, or biogeochemical tracers with seasonal and decadal drivers (insolation, circulation indices, anthropogenic forcing). Let p encode the realized driver history: seasonal phase, ENSO index, aerosols, etc. The state x is a field (PDE state) or a reduced vector (EOF/POD coefficients).

Absorbing neighborhoods. Energy inequalities give pullback absorbing balls in H^1 or L^2 : $d/dt \|x\|^2 \leq -\gamma \|x\|^2 + \beta(t)$, with $\beta(t)$ derived from past forcing magnitudes. Choose B_p as a ball with radius based on the past convolution of $\beta(t)$.

V as nowcasting metric. Compute $V(p,x)$ by simulating the process from a trailing history window and forming $S(p,t)=\Phi(t,\theta_{\{-t\}}p,B_{\{\theta_{\{-t\}}p\}})$. Evaluate V on reduced coordinates (first 10–50 EOFs). Low V means the present state is aligned with the current regime slice $\tau(t)=A_{\{\theta_t p\}}$.

Diagnostics.

- Level sets of V over leading EOF planes show regime basins.
- Drift sensitivity: jitter the last few weeks of driver data; tight V-bands $\Rightarrow$ robust “now-stability.”

Operational use.

- Early warnings: rising V precedes regime exits (e.g., stratification breakdown, monsoon onset shifts).
- Policy simulations: compare interventions by their ability to keep V in a green band without assuming long-run futures.
- Data assimilation: weight ensemble members by $\exp(-\lambda V)$ to privilege analyses near $\tau(t)$.

8.2 Epidemiology and operations

Setting. Nonstationary S(E)IR-type models with time-varying contact rates, behavior, seasonality, stochastic importations; or hospital/plant operations with schedule and demand drift. p carries the realized schedule/behavior history; x are compartment sizes or queue/work-in-process vectors.

Absorbing neighborhoods. For SIR with incidence $\beta(t)S I$ and removal γI , use Lyapunov-like totals (e.g., $I+\kappa R$) to obtain $d/dt E \leq -cE + \tilde{\beta}(t)$. Build B_p with radius from the past $\tilde{\beta}$ convolution. In operations, use linear/quadratic storage functions (workload Lyapunov) to bound WIP under realized inflow history.

V as situational risk. $V(p,x) \approx$ “distance from the present stable slice” $\tau(t)$ (low prevalence equilibrium or low-WIP manifold) using only the observed past interventions and arrivals. Forward attraction may fail if tomorrow’s policy changes; V still tells you how safe you are now.

Diagnostics.

- Heatmaps of V across (β_{now}, x) show how much margin remains.
- Horizon checks: if V_T stabilizes by $T \approx 2$ generation times (epidemics) or a few service times (queues), the absorber is credible.

Operational use.

- Trigger rules: enact NPIs or staffing changes when V crosses thresholds (green/yellow/red).
 - Adaptive control: short-horizon MPC minimizes V_T subject to cost; no prophecy needed beyond bounded near-term uncertainty.
 - Robust scheduling: pick shift patterns that keep V small under plausible demand drift envelopes.
-

8.3 Finance and policy regimes

Setting. Drift-prone, regime-switching dynamics for macro/markets: x collects filtered state (e.g., inflation gap, term-spread modes, risk factors); p is realized policy/flow/news history (rate path, balance-sheet actions, fiscal events).

Absorbing neighborhoods. Build B_p from realized-volatility envelopes and leverage/flow limits; for linear state-space filters $x' = A(t)x + B(t)u + \dots$, use Riccati/energy bounds to get $d/dt \|x\|^2 \leq -\gamma \|x\|^2 + \beta(t)$. In data-first setups, learn an empirical absorber as a hull of pullback images from bounded priors.

V as alignment to the current regime. $V(p, x)$ quantifies how far portfolios or macro states are from the present regime slice $\tau(t)$ (e.g., “tightening + disinflation corridor”). Low $V \rightarrow$ aligned exposures; rising $V \rightarrow$ regime slip risk.

Diagnostics.

- Regime maps: plot V over (duration, carry, FX beta) or principal components of factors; identify low- V corridors consistent with the realized policy path.
- Stress micro-futures: apply bounded 1–2 step shocks (rate bump, balance-sheet headline) and use the forward estimate $V(\theta_{\Delta} p, \Phi(\Delta, p, x)) \leq e^{-\alpha \Delta} V(p, x)$ to gauge fragility without full scenario trees.

Operational use.

- Risk budgeting: cap positions so that portfolio state stays within $V \leq \epsilon$ under daily drift envelopes.
- Rebalancing: prioritize trades that yield the greatest ΔV reduction per unit cost.
- Policy analysis: compare alternative rule paths by the size and stability of their low- V regions for the real economy state.

Cross-domain pattern. In all three arenas, $\tau(t)=A_{\{\theta_t p\}}$ is the “truth slice” defined by history; $V(p,\cdot)$ is the computable divergence to that slice. You monitor V to nowcast alignment, design actions to keep it small, and you do not need to assume anything about distant futures to make reliable, present-tense decisions.

9. Practical Guidelines and Pitfalls

9.1 Verifying absorption vs mistaking transients

What you want: a present-time family B_p that truly *pullback-absorbs* bounded sets—i.e., for large t , $\Phi(t, \theta_{\{-t\}p}, U) \subset B_p$ for any bounded U .

Checks that actually falsify mistakes

- Plateau test (radius vs horizon). Track $R(T) := \sup_{\{x \in U\}} \text{dist}(\Phi(T, \theta_{\{-T\}p}, x), 0)$ in your chosen norm. Absorption needs monotone stabilization: $R(T)$ nonincreasing beyond some T_0 and $|R(1.5T) - R(T)| \leq \epsilon \cdot R(T)$. If $R(T)$ wiggles or drifts upward, you’re seeing a transient or model error.
- Hull saturation test. Build $S_T := \Phi(T, \theta_{\{-T\}p}, U_{\text{grid}})$ with a coarse cover U_{grid} of a big ball. Increase cover density $\times 2$. If the minimal enclosing hull of S_T changes by $> \epsilon$ in radius or shape, absorption is not yet credible.
- Back-off inflation. After selecting B_p (ball/ellipsoid/hull), inflate by 5–15%. Re-run two additional horizons $T' > T$. If images still escape the inflated B_p , your absorber is underfit.
- Cross-section stability (PDEs). Check energy on nested domains: tail norms outside growing balls must decay. If tails re-energize at larger T , you only captured a transient front.
- Driver splice test. Replace a small *early* slice of the realized history with nearby histories (same statistics). A true absorber is insensitive to tiny, old splices; sensitivity indicates long memory $\rightarrow$ increase horizon or revisit dissipation bounds.

Common pitfalls

- Using a too-small U (or single nominal point) $\rightarrow$ false absorption.
- Estimating B_p from forward runs (starting now) $\rightarrow$ violates pullback logic.
- Ignoring parameter regions where local Lipschitz L blows up $\rightarrow V$ may under-weight real expansion.

9.2 Finite-window surrogates for infinite-horizon pullback

You will compute V with a finite window T . Do it so the surrogate V_T is trustworthy.

Choosing T and Δt

- Dwell-time heuristic. Take $T \approx 5\text{--}10\times$ the dominant contraction time $1/c$ (from energy inequality or estimated Lyapunov rate). For seasonal drivers, include $\geq 2\text{--}3$ full seasons.
- Stability of sup. Compute V_T and $V_{\kappa T}$ ($\kappa \approx 1.25\text{--}1.5$). Require $|V_{\kappa T}(p, x) - V_T(p, x)| \leq \epsilon_V$ for all x in a test mesh near τ . If not, extend T .
- Weight sanity. Ensure the cumulative Lipschitz integral $\sum L(\theta_{-s}p)\Delta t$ is accurate (no coarse Δt). Under-integration over-weights far horizons and can bias V_T downward.

Discretization and approximation

- Ellipsoidal push-forward. If $B_{\{\theta_{-t}p\}}$ is a ball and Φ is differentiable, approximate $S(p, t)$ by pushing the center and linearizing the radius (ellipsoid). This keeps distance queries cheap and reduces Monte-Carlo noise.
- Adaptive horizon. Stop increasing t once $W(p, t) \cdot d_t$ falls below ϵ_V for K consecutive steps (W is the weight, d_t the distance). Store the last “active” t as an audit trail.
- Two-sided error bars. Report V_T with a band: (i) upper from sampling/coarse grids (distance overestimation), (ii) lower from linearized ellipsoids (distance underestimation). If the band straddles a decision threshold, refine.

Surrogates that break

- Using a fixed, small T across regimes with very different time scales.
- Letting $L(p)$ be estimated from calm periods, then applying it during spikes (choose a conservative envelope $\hat{L} \geq L$ over a trailing window).

9.3 When forward attractors remain the right tool

Pullback logic shines when the future drifts or is unknown. Still, forward/uniform attractors can be preferable:

Use forward attractors when

- Stationary or ergodic futures. The driver is statistically time-invariant, or you commit to a fixed scenario (e.g., periodic forcing). Then forward attraction to $A_{\theta_t p}$ delivers genuine asymptotics.

- Design for convergence. Your objective is *eventual* convergence to a target operating regime, not just present alignment. Control and model reduction often need forward attractivity to certify steady performance.
- Hyperbolic structure available. Exponential dichotomies and Lyapunov–Perron yields invariant manifolds/graphs with forward exponential tracking; invaluable for normal forms and rigorous bifurcation analysis.
- Global summaries. The uniform/global attractor gives a time-independent compact set summarizing all long-time bounded behavior—useful for cataloging regimes, even if $\tau(t)$ slices are your day-to-day metric.

Red flags (don't rely on forward)

- Evidence of structural drift (seasonality shifts, secular trends, changing constraints).
- Policy or schedule that is re-optimized frequently (operations/markets).
- High sensitivity of forward limits to small, recent changes in the driver.

Practical compromise

- Hybrid certification. Use pullback V for nowcasting and safety; use forward attractors/manifolds for planning under clearly stated scenarios. Keep both in view; report which assumption each certificate depends on.
- Trigger for switch. Maintain a drift diagnostic (e.g., KL divergence or a CUSUM test on recent driver increments). If drift < threshold over long windows, elevate forward analyses; if drift spikes, fall back to pullback-only decisions.

Bottom line.

Pullback attracts certify “what is stably real now” ($\tau(t)$); forward attracts certify “where you will go” under a declared future. Choose by what you *know* (history) versus what you're *willing to assume* (future).

10. Conclusion

10.1 What pullback Lyapunov functions certify—and what they do not

They certify

- Present-time stability: a small $V(p, x)$ means x lies close to the pullback attractor slice A_p (the “truth slice” $\tau(t)$) determined only by realized history.

- Lyapunov decay without prophecy: along forward time, $V(\theta_t p, \Phi(t, p, x)) \leq e^{-\alpha t} V(p, x)$; along pullbacked trajectories, $V \rightarrow 0$.
- Separation: $V(p, x) = 0 \iff x \in A_p$, and locally V is equivalent to $\text{dist}(x, A_p)$.
- Operability: V can be computed from absorbing neighborhoods B_p , pullback images $S(p, t)$, and Lipschitz envelopes L , all estimable from bounds or data.

They do not certify

- Eventual convergence to moving futures: V does not guarantee $x(t) \rightarrow A_{\theta_t p}$ as $t \rightarrow \infty$ when the environment drifts.
- Global optimality or safety beyond the absorber: outside regions where B_p and L are credible, the certificate can be void.
- Manifold geometry: V witnesses alignment but does not identify invariant manifolds or normal forms (that is the Lyapunov–Perron/dichotomy domain).
- Scenario claims: any statement that depends on unobserved future drivers lies outside pullback certification.

10.2 The virtue of “past $\rightarrow$ present” realism

- Epistemic honesty: Pullback logic asks only what history has already revealed. It replaces “assume a future” with “condition on the past,” avoiding false certainty in drifting worlds.
- Operational clarity: By framing $\tau(t) = A_{\theta_t p}$ and measuring $D(q(t) \parallel \tau(t)) = V(p_t, q(t))$, we obtain a scalar, auditable nowcasting metric for alignment—green/yellow/red thresholds that can drive decisions today.
- Robustness to drift: Because the construction weights distances with accumulated local expansivity $L(\theta_{-s} p)$, it naturally discounts far history and stabilizes against recent noise.
- Composability: V slots into ISS-style bounds, MPC cost functions, barrier designs, and filter weights (e.g., ensemble reweighting by $\exp(-\lambda V)$)—a universal glue between analysis and control.

10.3 Next steps: numerics, data-driven drivers, open benchmarks

- Numerical playbook.
 1. Derive or learn dissipative bounds to define B_p .
 2. Estimate $L(p)$ conservatively on a sliding window.

3. Compute V_T with adaptive horizons until the sup stabilizes; report bands (sampling vs linearization).
 4. Plot level sets of $V(p, \cdot)$ and deploy thresholds for operations.
- Data-driven drivers.
 - Build θ from observed sequences (seasonal indices, schedules, policies).
 - Use empirical hulls for B_p and finite-difference Jacobians for L .
 - Maintain drift diagnostics (CUSUM/KL on driver increments) to decide when to privilege pullback over forward analyses.
 - Benchmarks and sharing.
 - Curate open pullback testbeds (time-varying ODEs, seasonally forced PDE surrogates, queue/epidemic models) with reference B_p , L , and vetted V code.
 - Standardize finite-window validation (plateau tests, horizon stress-tests, jitter ensembles) so certificates are comparable across domains.
 - Publish alignment dashboards: live $V(p_t, \cdot)$ maps, robustness bands, and controller set-points to make past $\rightarrow$ present stability legible to practitioners.

Bottom line. Pullback Lyapunov functions give a rigorous, computable certificate of alignment with $\tau(t)$ now. They trade prophecy for realism, pair naturally with control and inference, and invite a practical ecosystem—numerical recipes, data-driven drivers, and open benchmarks—that makes nonautonomous stability actionable in the world we actually observe.

References

- Kloeden, P. E. (2000). *A Lyapunov function for pullback attractors of nonautonomous differential equations*. Electronic Journal of Differential Equations, Conference 05, 91–102. ejde.math.txstate.edu+1
- Kloeden, P. E., & Rasmussen, M. (2011). *Nonautonomous Dynamical Systems*. Mathematical Surveys and Monographs, Vol. 176, American Mathematical Society. bookstore.ams.org+1
- Crauel, H., & Flandoli, F. (1994). *Attractors for random dynamical systems*. Probability Theory and Related Fields, 100, 365–393. <https://doi.org/10.1007/BF01193705>
link.springer.com+1
- Wang, B. (2012). *Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems*. Journal of Differential Equations, 253(5), 1544–1583. sciencedirect.com+2sciencedirect.com+2
- Cheban, D. N. (2015). *Global Attractors of Nonautonomous Dynamical and Control Systems* (2nd ed.). World Scientific. ISBN 978-9814619820. [ResearchGate](http://ResearchGate.com)+2[Amazon](http://Amazon.com)+2
- Caraballo, T., & Han, X. (2016). *Applied Nonautonomous and Random Dynamical Systems* (lecture notes/monograph). (Overview emphasizing pullback attractors.) idus.us.es
- Caraballo, T., Kloeden, P. E., & Schmalfuß, B. (2003). *Pullback attractors of nonautonomous and stochastic dynamical systems*. Nonlinear Analysis: Theory, Methods & Applications, 52(2), 271–291. (See also related results on multivalued processes.)
link.springer.com
- Cui, H., Kloeden, P. E., Langa, J. A., & Yang, M. (2017). *Uniform attractors for non-autonomous random dynamical systems*. Journal of Differential Equations, 263(6), 3443–3470. sciencedirect.com
- Samproгна, R., & Caraballo, T. (2018). *Pullback attractor for a dynamic boundary non-autonomous problem with infinite delay*. Discrete & Continuous Dynamical Systems – B, 23(9), 3891–3913. aimsciences.org
- Flandoli, F. (notes). *An introduction to random dynamical systems for climate* (tutorial notes; includes pullback/random attractor background). users.dma.unipi.it+1

(The core items for the Lyapunov construction are Kloeden (2000) and Kloeden–Rasmussen (2011); the remaining references provide extensions to random/pullback settings, existence criteria, and applications.)